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Quantum Mechanics and the Local Observer

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A notion of local observer inspired by the work of Segal is introduced here in the Hilbert space theory of quantum mechanics. The local observer finds a mathematical place in the Hilbert space through local negation or complementation. A logicomathematical theory of local negation is presented and its implications for quantum logic and the problem of measurement are discussed. The setting is constructivist mathematics and the main result of the paper states that the introduction of a local observer implies the nonorthocomplementability of the whole Hilbert space even in the finite-dimensional case. Making a mathematical place for the observer (the "projector") thus modifies the structure of the observables or the system of the projections, in accordance with a nonclassical theory of quantum-mechanical measurement.

The problem of the observer is a long-standing concern of quantum mechanics and attempts at formulating a coherent theory of the observer have been apparently unsuccessful. In this paper, I want to describe a mathematical notion of the observer inspired by the work of I. E. Segal (1976) in his chronogeometric theory of relativity.

1. LOCAL OBSERVER

Segal defines the observer in the following way: let M be a globally hyperbolic (causal) manifold. Then a prefactorization is a pair (S, φ) where S is a C^{∞} manifold, φ is a diffeomorphism of $T \times S$ onto M, and T is a real interval, having the properties that

1. $\forall x \in S$, $t \to \varphi(t, x)$ is a timelike arc in M
2. $\forall t \in T$, $x \to \varphi(t, x)$ defines a spacelike submanifold of M.

For two prefactorizations (S, φ) and (S', φ') to be equivalent there must

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exist diffeomorphisms f and g of R^1 onto R^1 and S onto S' such that f is orientation-preserving, and

$$\varphi(f \times g)^{-1} = \varphi'$$

A factorization is simply an equivalence class of such prefactorizations. A local observer in that context is a prefactorization $(U, \varphi,)$ on M with $\varphi_1 = \varphi|T \times U$, where T and U are connected open subsets of R^1 and S; here T is only diffeomorphic to R' ($\varphi|T \times U$ means the restriction of φ to $T \times U$).

Local observers, in Segal's theory, may also have the properties of being metric, homogeneous, physical, or covariant, all properties associated with the spaces on which the local observer is defined. In Segal's words, "local observer" is the mathematical counterpart of the physical concept of "local Lorentz frame." I am interested here only in one part of Segal's theory, his notion of local observer. If we start from the fact that an n-dimensional manifold is a topological space which is locally homeomorphic to R^n, we can ask ourselves if a characterization of the observer as "local observer" is at all possible in QM; if there could be a treatment of the observer in QM similar to Segal's notion.

2. HILBERT SPACE

The usual presentation of QM requires the analytical apparatus of Hilbert space as a linear vector space with complex coefficients; among all linear manifolds that constitute a Hilbert space, the closed ones or the subspaces are of special interest for physics (i.e., QM here), since notions like orthogonal vectors, orthogonal complements, projections, etc., can be defined on them. It is a well-known fact that not all linear manifolds are closed[1] and that the set of all linear subsets of the infinite-dimensional Hilbert space is not orthocomplementable[2]: it is this result which I want to exploit, keeping in mind that a Hilbert space is a metric and a topological space. The interesting fact about Hilbert space from a physical point of view is that it permits the definition of orthogonality

$$(f, g) = 0$$

written $f \perp g$; the orthogonal complement of f, $f^{\perp}$ obeys the Boolean rule

[1] Cf. Halmos (1957), p. 22.
[2] Cf. Jauch (1968), p. 122.

$f^{\perp\perp} = f$ and $f^{\perp}$ forms a subspace of $\mathcal{H}$. For QM, it is important to notice that there is a bijection between subspaces and projections, i.e., the linear operators E such that $EE^* = E$ for E^* the adjoint of E defined by $(E^*)^* = E$ (if $E^* = E$, then E is a self-adjoint or Hermitian operator). The spectral theorem states that there is a bijection between self-adjoint operators and spectral measures on (the Borel sets of) the real line R^1 and the von Neumann "dogma" states that there is a bijection between self-adjoint operators and the observables of QM.[3] Let us look at the orthogonal complement: we have seen that $f^{\perp\perp} = f$; consequently, the orthogonal complement corresponds to the orthocomplement $(a^-)^- = a$ of a Boolean lattice, where $\leqslant$ corresponds to $\rightarrow$, a^- to $\sim a$, $a \cap b$ to $a \wedge b$, and $a \cup b$ to $a \vee b$. Orthocomplementation induces an involutive antiautomorphism $(a^*)^* = a$ on the field of a vector space. It is such an antiautomorphism which yields Gleason's important theorem (1957) stipulating that any probability measure $\mu(A)$ on the subspaces of $\mathcal{H}$ has the form

$$\mu(A) = \mathrm{Tr}(WP_A)$$

where Tr means $\mathrm{Tr}\, X = \Sigma_r(\varphi_r, X\varphi_r)$ for any complete system of normalized orthogonal vectors φ_r, P_A denotes the orthogonal projection of A, and W is a Hermitian operator which satisfies

$$W > 0, \qquad \mathrm{Tr}\, W = 1 \quad \text{and} \quad W^2 \leqslant W$$

Other spaces, like Banach spaces, which lack the restriction of orthogonality, do not seem to be suited to the needs of QM.

3. LOCAL NEGATION

I want to introduce the observer in QM with the help of a notion of local negation: local negation induces local complementation which provides the "location" of the local observer. I think of local negation in analogy with local notions in mathematics, especially topology; e.g., for metrization, local finiteness means that for a family C of subsets of a topological space each point of the space has a neighborhood which intersects only a finite number of members of C. Local negation can be

[3] Von Neumann's dogma has been challenged in 1952 by Wick, Wightman, and Wigner, who introduced superselection rules showing that there exist Hermitian operators that do not correspond to observables; on the other side, Park and Margenau argue that there are observables, for example, the noncommuting x and z components of spin, which are not represented by Hermitian operators. Cf. my paper "The Use of the Axiomatic Method in Quantum Mechanics" in *Philosophy of Science*, **38**(3), 429–437 (1971).

thought as the logical counterpart of the relative complement in topology, while Boolean negation ($\sim\sim a = a$) corresponds to the absolute complement.

I come now to the logicomathematical theory of local negation. The setting is constructivist mathematics. There are various approaches to the constructivization of classical mathematics, intuitionism (Brouwer), predicativism (Weyl, Lorenzen), Russian constructivism, numerical constructivism à la Bishop, etc.... Here I borrow some notions from Brouwer's intuitionism and Bishop's constructivism, because they fit particularly well in the scheme of local negation (or complementation). Neither Brouwer nor Bishop has developed such a theory of local negation; Bishop, for example, is led in his theory to distinguish between negation and complementation. But there are various constructivizations of the theory of Hilbert spaces that could serve as the general context of the notion of local negation. Von Neumann (1950) himself was not unaware of those constructive aspects of mathematics in his proof of the existence of a complete orthonormal set in any (separable) linear space; von Neumann gives two versions of his proof, one by construction and one by set-theoretic means. I begin by stating some notions.

Definition 1. The domain D is the collection of mathematical properties or assertions pertaining to a mathematical theory—Brouwer used here the concept of species instead of set, because he wanted to emphasize the intensional character of the notion of mathematical property in contrast with notions like the definition of extensional equality:

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

Definition 2. The exterior E is the local negation or complement of domain D; in symbols

$$E^D =_{\mathrm{df}} \neg D$$

for this notion of local complement, $\neg\neg D = D$ does not obtain.

Definition 3. An effinite sequence is a sequence (of natural numbers, for example) that has an initial bound (0, for example) but no terminal bound (ω, for example)—Brouwer called such sequences "infinitely proceeding sequences" in order to stress their potential infinity. Finite sequences are sets.

In the present context, D is the domain of closed linear manifolds of a Hilbert space $\mathcal{H}$ and E its local complement $E \in \mathcal{H} - D$. This is the usual topology of sets; I assume that D and E are not sets, but effinite sequences; then set complementation does not obtain and D and E are to be considered as effinite sequences (of singular instances) of mathematical assertions (and of their negations). Now we need an evaluation of those assertions in terms of 0 and 1. The idea here, which is of frequent use in intuitionistic mathematics, is to associate to every assertion (or closed formula) a natural number which calculates it, in the sense that it gives the formula an effective assignment in the sequence of natural numbers; we have thus a mapping

$$\xi: N \to N$$

which sends every mathematical assertion into a natural number once it is evaluated by a natural number or a sequence of natural numbers (this mapping is often called a complementary mapping in intuitionistic literature).

A model for local negation is a quadruple $M = \langle D_M, E_M, \mathbb{C}_M, \varphi_M \rangle$, where D and E are defined as above, $\mathbb{C}$ is a relation or order or superposition for domains $D_{M_0}, D_{M_1}, \ldots, D_{M_n}$ and φ a map

$$\varphi_M: \text{Form} \to (0, 1)$$

which evaluates formulas (of a language) in the following manner:

1. $\varphi_M(A)[n] = 1$, iff $A \in D_{M_0}$
2. $\varphi_M(\neg A)[n] = 1$, iff $\neg A \in E^D M_0$
3. $\varphi_M(A \wedge B)[n \times m] = 1$, iff $A \in D_{M_0}$ and $B \in D_{M_0}$
4. $\varphi_M(A \vee B)[n + m] = 1$, iff $A \in D_{M_0}$ or $B \in D_{M_0}$
5. $\varphi_M(A \overset{\text{loc}}{\to} B)[n^m] = 1$, iff $A \in D_{M_0}$ is transformed exponentially (or continuously) into $B \in D_{M_0}$
6. $\varphi_M(\exists x A x)[n + m + l \ldots] = 1$, iff $\Sigma A_n \in D_{M_0}$
7. $\varphi_M(\forall x A x)[n \times m \cdots \times l] = 1$, iff $\Pi A_n \in D_{M_0}$
8. $\varphi_M(\Xi x A x)[n \times m \times l \ldots] = 1$, iff $\Pi A_{n \ldots} \in D_{M_0}$

Remarks. (a) Although D and E are not sets, we have the usual symbol $\in$ for convenience; (b) the universal quantifier $\forall x$ is meant to apply only to finite sets while the new quantifier Ξx applies to effinite sequences (the n, m, l are natural numbers); (c) clause 5 shows that local implication can be thought of as an arc or a closed topological path in a locally connected space.

With this machinery, it is possible to formulate the notion of local complementation which follows from the notion of local negation.[4]

[4] For more details on the theory of local negation, I refer to my "Intuitionistic Logic and Local Mathematical Theories" in *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* **23**(5), 411–414 (1977).

4. LOCAL COMPLEMENTATION

I have exposed thus far a logical theory of local negation (or complementation). Here I apply this notion to the theory of Hilbert spaces. Consider the Hilbert space as a metric and a topological space; D is in this case the sequence of subspaces of the Hilbert space and E is obtained by local complementation; E is the "location" of the local observer. We shall see that the Hilbert space can make room for a notion of local observer: the observer becomes the (local) complement of the observable, i.e., the closed linear manifolds of the Hilbert space—of course the whole Hilbert space contains all bounded linear transformations (defined on open subsets) and is therefore not orthocomplementable. But here we obtain nonorthocomplementability in a different way. [Remember that in a finite-dimensional space, every linear manifold is closed; cf. Halmos (1957).]

> *Theorem.* The Hilbert space admits the observer through local negation (or complementation)—that is, we do not have orthocomplementation on the whole Hilbert space even in the finite-dimensional case.

Before we prove the main theorem and in order to make the proof easier, we prove some lemmas and give a few definitions.

> *Lemma 1.* The sequence of closed linear manifolds in a Hilbert space has a local complement.

Proof. Let $\mathcal{H}$ be an n-dimensional Hilbert space and let $F^{\perp}$ be the sequence of closed linear manifolds (or subspaces) $f^{\perp}$ of $\mathcal{H}$. Set $F^{\perp} = F^{-}$, the closure of all $f^{\perp}$. One can now define the local complement $E(F^{-})$ or F^{+} of F^{-} such that $F^{+} \in \mathcal{H} - F^{-}$ and it is an open manifold of $\mathcal{H}$. ■

From the topology, we pass to the metric of $\mathcal{H}$.

Definition 4. A is a subsequence of B, iff $A \subset B$.

Remark. The usual concept of limit is applicable to sequences and subsequences with minor modifications, which are not essential here.

Definition 5. A subsequence A is local (or located),[5] if the distance

$$\forall x \in \mathcal{H}\left[\rho(x, A) \equiv \inf\{\rho(x, y) : y \in A\}\right]$$

from x to A exists.

> *Lemma 2.* A local subsequence A has an open complement.

[5] This notion of located subset has been introduced by Brouwer. E. Bishop has put it to use in his *Foundations of Constructive Analysis*, McGraw-Hill, New York (1967), p. 82.

Proof. The metric complement $-A$ of a subsequence A is the sequence

$$-A \equiv \{x: x \in \mathcal{H}, \rho(x, A) > 0\}$$

which is open, since

$$\forall x, y \in \mathcal{H}[\rho(x, A) \leqslant \rho(x, y) + \rho(y, A)] \qquad \blacksquare$$

The observer could find here a topological and a metric location as the local complement of the closed sequence of subspaces of $\mathcal{H}$. In order to further constructivize this result, I introduce the topological boundary operator b (which is to be interpreted as the boundary between the observable—or the observed—and the observer).

Definition 6. The boundary of a subsequence A of a topological space X is the sequence of all points x which are interior to neither A nor $X - A$.

Proof of the Theorem. We have the relations

$$E = \neg D - b(E)$$

and

$$D = \neg E \cup b(D)$$

thus

$$\neg\, D(\mathcal{H}) = E(\mathcal{H}) \cup b(E(\mathcal{H}))$$

The interior of E, i.e., E^0, is the complement of the closure of the complement of E and is thus open; also we have

$$E = E^0$$

For any x, $D(\neg x)$ means that $x \in E$. So for some $a \in \mathcal{H}$, we have

$$E(a) = D(\neg a) - b(D(\neg a))$$

on the other hand, the closure of D, i.e., D^-, implies that

$$b(D(a)) = a^- \cap (D - a)^-$$

hence

$$a^- = a \cup b(D(a))$$

and

$$\neg\, a \in E(\mathcal{H}) = a \in D(\mathcal{H}) \cup b(a \in D(\mathcal{H}))$$

and

$$a \in D(\mathcal{H}) = \neg a \in E(\mathcal{H}) - b(E(\mathcal{H}))$$

which shows that E is disjoint from its boundary, that is, it is open and consequently the whole Hilbert space $D(\mathcal{H}) \cup E(\mathcal{H})$ is not orthocomplementable, since local complementation excludes $(a^-)^- = a$.[6] ∎

Remark. The effect of abandoning orthocomplementation amounts to adopt an indefinite metric which may, in fact, be more convenient for some physical theories (e.g., quantum field theory).

The significance of the theorem lies in the fact that a mathematical location is secured for the observer. The presence of the (local) observer "opens up," through the local open complement, the sequence (classically, the set) of closed linear manifolds of the Hilbert space. The implications of the theory of local complementation are far-reaching, if one considers quantum logic and the problem of measurement in QM. From the local point of view here discussed, it is clear that the underlying logic of QM is not classical, but is of a constructivist variety close to intuitionist logic with a pseudocomplement. Further, the theory of local observer offers a new solution to the various paradoxes of measurement. In the next two sections, I briefly sketch the impact of the notion of local observer on those problems.

5. QUANTUM LOGIC

Recently, many workers in the field of the foundations of QM, Finkelstein, Putnam, Bub, and Demopoulos among others, have advocated the idea of a quantum logic. The result of Kochen and Specker (1967), which is closely linked with Gleason's theorem, is seen as the final blow to (noncontextual) hidden-variable theories. Kochen and Specker have shown that in a space of more than two dimensions, there is no two-valued homomorphism $h\colon A \to A'$ from the algebra A of partial operations on

[6]Orthocomplementation requires that

$$(a^-)^- = a, \qquad a^- \cap a = \varnothing, \qquad \text{and} \qquad a \leqslant b \leftrightarrow b^- \leqslant a^-$$

compatible observables to a commutative Boolean algebra A'. The partial algebra of quantum mechanical propositions is consequently not embeddable in a Boolean algebra [for a recent exposition, see Bub (1976)]. A partial algebra is a set A with a binary relation of compatibility $\downarrow$ which is symmetric and reflexive, but not transitive; it is also closed under the operations of addition and multiplication from $\downarrow$ to A and closed under the scalar product; we have

$$(1)\ \downarrow \leqslant A^2 \qquad (2)\ a \downarrow a \qquad (3)\ \forall a, b \in A(a \downarrow b \rightarrow b \downarrow a)$$
$$(4)\ (a+b) \downarrow c, \qquad ab \downarrow c; \qquad \lambda a \downarrow b$$

and the unit element; it is a partial algebra because its operations are partial —an observable does not necessarily possess a value for each of its states and it is a partial Boolean algebra since the set of idempotent elements $a \cdot a = a$ of the partial algebra constitutes a Boolean algebra with

$$a \wedge b = a \cdot b, \qquad a \vee b = a + b - a \cdot b, \qquad a^{-} = 1 - a \quad \text{and} \quad (a^{-})^{-} = a$$

Instead of a partial Boolean algebra, one can construct a partial Heyting or pseudo-Boolean algebra in which the relative complement replaces the Boolean complement.[7] For a lattice B, an element c of B is the pseudocomplement of a relative to b, if it is the largest element such that

$$a \cap c \leqslant b$$

it is thus the largest open subset different from a,

$$a \leqslant \neg\neg a$$

for $\neg a$, the local negation of a. Such a treatment would permit still a further "constructivization" of QM.

6. THE PROBLEM OF MEASUREMENT

The topological theory of the local observer bears also on the problem of measurement, and the notion of local observer could help clarify some baffling problems of measurement in QM. Let us discuss briefly the Einstein–Podolsky–Rosen paradox. Let I and II be two systems which

[7]Bub (1976) draws here upon results of MacNeill and Petersma and states that a Heyting algebra as a distributive lattice is embeddable in a Boolean algebra; but it should be noted that the result is valid only for complete Heyting algebras enriched with additional algebraic structure, i.e., a canonical extension in this case.

eventually will interact; the states of the two systems are described in a two-dimensional vector space and $\varphi\pm$ and $\psi\pm$ represent a complete orthonormal set of vectors for systems I and II, respectively. The pure state of the joint system is defined by

$$\Phi = 2^{-1/2}[(\varphi_+ \otimes \psi_+) + (\varphi_- \otimes \psi_-)]$$

where Φ is the probability and $\otimes$ the tensor product; probability is defined by

$$\text{prob}(a_k) = \sum_k |\langle \varphi_{k,r} | \psi \rangle|^2$$

for a_k the eigenvalues of the operator $\mathscr{A}$ which corresponds to the observable A; $|\psi\rangle$ is the normalized state vector of the system and the $|\varphi_{k,r}\rangle$ are the normalized eigenvectors of $\mathscr{A}$. After spatial separation, system I is in the state $\varphi\pm$ with probability 1/2; for system II, we have

$$\text{prob II}(\psi\pm) = 1/2$$

since

$$\text{prob I}(\varphi\pm) + \text{prob II}(\psi\pm) = 1/2 + 1/2 = 1 = \Phi$$

The logic of the paradox implied that the state of system II could be determined from the state of system I without having direct access to system II—in Einstein's view, there was an element of reality here. The paradox is usually solved by the simple remark (made originally by Bohr) that after the measurement, the system (I + II) is in a composite state ($W_1 \otimes W_2$) or in a mixture and not in a single state W. Measurement in a way does not conserve the eigenvectors (and eigenvalues) of each of the system. Some, like Wigner, introduce at this very point the consciousness of the observer; others invoke "the state of the knowledge of the state," which ultimately would explain the probabilistic structure of the quantum world. From my point of view, the observer is not endowed with any particular (mysterious) property of consciousness or knowledge—it is a local observer of which we have only a mathematical description or localization. The boundary between the observed and the observer could be seen as a von Neumann's cut, but I prefer to interpret it in a purely topological sense.

Let us mention as a further illustration J. S. Bell's theory (1965) of local hidden variables ("local" is taken here in the sense of spatial separation or relativistic causality). The much-debated Bell inequality concerns the measurement of spin components A, B, C of n particles and could be written

$$n[A^+B^+] \leqslant n[A^+C^+] + n[B^+C^+]$$

Experiments in QM violate (most of the time) this inequality and some invoke as a solution of this new paradox principles of wholeness or integrity of the quantum world [cf. the works of Bohm or d'Espagnat (1979)]. Others have used the many-worlds interpretation of Everett without noticing its inconsistency, which can be shown quite easily: in Everett's formulation, the wave function ψ must take all its values in the complete ramification of the "universal wave equation," that is, it must take $2^{\aleph_0}$ values, but there is certainly not more than a denumerable infinity of possible measurements, that is $\aleph_0$; there is no bijection between $2^{\aleph_0}$ and $\aleph_0$, and in view of Everett's thesis about the isomorphism (or homomorphism) between the formalism and the interpretation of QM, this suffices to refute the many-worlds interpretation. As in the case of the Einstein–Podolsky–Rosen paradox, one could use again the standard Bohr answer: measurement or the observer modifies the original phenomenal situation in such a way that, for example, orthogonal probability measures (or measures on orthogonal subspaces) are not conserved, since the effect of measurement (or local observer) is to "open up" the Hilbert space of the observables through the admission of a local open complement of the closed linear manifolds of the Hilbert space.

7. CONCLUSION

In relativity theory, the local observer is a local Lorentz frame of reference which is part of the objective picture of the physical universe described by the theory. For QM, a theory of the physical observer whould have to take into account some kind of electromagnetic interaction (which I would call "projector" in analogy with the notion of projection) between the observer and the observed system as Geoffrey Chew has pointed out. I have limited myself to a mathematical description of the "location" of the observer as the local complement of the set of observables in the Hilbert space of QM; the fact that, in the spirit of Segal's theory, the mathematical description of the local observer defines it as a prefactorization in relativity and as an open subset or an open submanifold of a Hilbert space in QM does not preclude a further characterization of the observer in terms of its physical attributes or interactions. The securing of a place for the observer in the Hilbert space of QM is only a first step in that direction.

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